

CONCENTRATION OF THE RATIO BETWEEN THE GEOMETRIC AND ARITHMETIC MEANS

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ABSTRACT. We explore the concentration properties of the ratio between the geometric mean and the arithmetic mean, showing that for certain sequences of weights one does obtain concentration, around a value that depends on the sequence.

1. INTRODUCTION.

This paper is motivated by the article [GluMi] by E. Gluskin and V. Milman, who considered the variant $\prod_{i=1}^n |y_i|^{1/n} \leq \sqrt{n^{-1} \sum_{i=1}^n y_i^2}$ of the AM-GM inequality in the equal weights case. Roughly speaking, they showed that the ratio $\prod_{i=1}^n |y_i|^{1/n} / \sqrt{n^{-1} \sum_{i=1}^n y_i^2}$ is bounded below by 0.394 asymptotically in n and with high probability, where probability refers to Haar measure on the euclidean unit sphere \mathbb{S}_2^{n-1} . Thus, the geometric and arithmetic means are comparable quantities on “large sets”, provided n is large.

For the most part, we focus on the standard AM-GM inequality $\prod_{i=1}^n |x_i|^{\alpha_{i,n}} \leq \sum_{i=1}^n \alpha_{i,n} |x_i|$, where $\alpha_{i,n} > 0$ and $\sum_{i=1}^n \alpha_{i,n} = 1$. We shall see that in some cases the concentration of measure phenomenon does take place: For certain sequences of weights, which include the equal weights case, the GM-AM ratio is “almost constant”, with probability approaching 1 as $n \rightarrow \infty$. Here probability means normalized area over a suitable unit sphere (a level set of the arithmetic mean $f(x) = \sum_{i=1}^n \alpha_{i,n} |x_i|$) which depends on the weights. We prove that the GM-AM ratio cannot concentrate around values larger than $e^{-\gamma} \approx 0.5615$, where γ is Euler’s constant. On the other hand, for every $t \in [0, e^{-\gamma}]$ it is possible to find a family of weights such that concentration takes place around t . In particular, when all weights are equal (to $1/n$ for $n = 2, 3, \dots$) there is concentration around $e^{-\gamma}$, a result to some extent anticipated in [Gl, Theorem 5.1].

For completeness, we also study Gluskin and Milman’s equal weights modification of the GM-AM ratio: $\prod_{i=1}^n |y_i|^{1/n} / \sqrt{n^{-1} \sum_{i=1}^n y_i^2}$. Here the natural choice of probability is the uniform measure P_2^{n-1} on $\mathbb{S}_2^{n-1} := \{\|y\|_2 = 1\}$, where $\|y\|_2 := \sqrt{\sum_{i=1}^n y_i^2}$ is the ℓ_2^n norm on \mathbb{R}^n . For $n \gg 1$ and on \mathbb{S}_2^{n-1} , the preceding ratio concentrates around $\sqrt{2} \exp[\Gamma'(1/2)/(2\Gamma(1/2))] \approx 0.5298$.

The method of proof used here is essentially the same as in [GluMi]: Compute the s -moment of $\prod_{i=1}^n |x_i|^{\alpha_{i,n}}$ and then use Chebyshev’s inequality. The differences lie in the fact that we consider the usual AM-GM inequality rather than a variant of it, we do not restrict

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ourselves to the equal weights case, and we optimize over s to obtain concentration results rather than lower bounds.

The statistical properties of the ratio between the geometric and the arithmetic means, in addition to its mathematical interest, may be relevant in those areas where this ratio is used as a measure of homogeneity or information, such as, for instance, radar imaging (cf. [BMTE], [Wo]).

2. DEFINITIONS AND RESULTS.

Given $n \geq 2$, weights $\alpha_{i,n} > 0$ with $\sum_{i=1}^n \alpha_{i,n} = 1$, and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, the AM-GM inequality states that

$$(2.0.1) \quad \prod_{i=1}^n |x|_i^{\alpha_{i,n}} \leq \sum_{i=1}^n \alpha_{i,n} |x|_i.$$

Assume $x \neq 0$. Since the middle term of

$$(2.0.2) \quad 0 \leq \frac{\prod_{i=1}^n |x|_i^{\alpha_{i,n}}}{\sum_{i=1}^n \alpha_{i,n} |x|_i} \leq 1$$

is a positive homogeneous function of degree zero, it is constant on the rays tv , where $t \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Thus, we may fix any suitable t and then select $v \in \{\sum_{i=1}^n \alpha_{i,n} |x|_i = t\}$ “at random”, that is, uniformly over this level set. Observe that the notion of random choice we use depends on the sequence $\alpha_n := (\alpha_{1,n}, \dots, \alpha_{n,n})$. Let us renormalize the weights by setting

$$(2.0.3) \quad a_{i,n} := n\alpha_{i,n}.$$

Define the unit ball in \mathbb{R}^n associated to $a_n := (a_{1,n}, \dots, a_{n,n})$ by $\mathbb{B}_{a_n}^n := \{\sum_{i=1}^n a_{i,n} |x|_i \leq 1\}$, and denote the corresponding norm and unit sphere by $\|\cdot\|_{a_n}$ and $\mathbb{S}_{a_n}^{n-1} := \{\sum_{i=1}^n a_{i,n} |x|_i = 1\}$ respectively. In the case of equal weights, $a_{i,n} = 1$, $\|\cdot\|_{a_n}$ is just the ℓ_1 -norm $\|\cdot\|_1$, $\mathbb{B}_{a_n}^n$ the cross-polytope \mathbb{B}_1^n , and $\mathbb{S}_{a_n}^{n-1}$ the ℓ_1 -unit sphere \mathbb{S}_1^{n-1} . For arbitrary a_n , $\mathbb{B}_{a_n}^n$ and $\mathbb{S}_{a_n}^{n-1}$ are simply linear images of \mathbb{B}_1^n , and \mathbb{S}_1^{n-1} .

To select $v \in \{\sum_{i=1}^n a_{i,n} |x|_i = t\}$ at random, by homogeneity we can just set $t = 1$, and then use the normalization of area on $\mathbb{S}_{a_n}^{n-1}$. Denote by \mathcal{H}^s the s -dimensional Hausdorff measure on \mathbb{R}^n . We adopt the convention that \mathcal{H}^s is defined via the Euclidean ℓ_2 metric, and normalized by the factor $\pi^{s/2}/\Gamma(1 + s/2)$, so if $s = d$ is a positive natural number, the cube of sidelength 1 has Hausdorff d measure 1. A consistent choice of normalizing factor is required, for instance, to use Fubini’s Theorem, or more generally, the coarea formula. We denote by $P_{a_n}^{n-1}$ the uniform probability on $\mathbb{S}_{a_n}^{n-1}$, that is $P_{a_n}^{n-1} = c\mathcal{H}^{n-1}$, where c is chosen so that $P_{a_n}^{n-1}(\mathbb{S}_{a_n}^{n-1}) = 1$. Occasionally, and for simplicity, we will utilize absolute value signs to denote volumes and areas.

The first three results deal with families of uniformly bounded renormalized weights. The fourth considers a sequence with some weights approaching ∞ , which yields concentration around zero.

Theorem 2.1. *Suppose that there exist an M such that $a_{i,n} \leq M$ for all $n \geq 2$ and all $i = 1, \dots, n$. Let γ be Euler's constant, let $k, \varepsilon > 0$, and let $P_{a_n}^{n-1}$ be the uniform probability on $\mathbb{S}_{a_n}^{n-1}$. Then there exists an $N = N(k, \varepsilon)$ such that for every $n \geq N$, the ratio of the geometric mean over the arithmetic mean on $\mathbb{S}_{a_n}^{n-1}$, given by $n \prod_{i=1}^n |x_i|^{\alpha_{i,n}}$, satisfies*

$$(2.1.1) \quad P_{a_n}^{n-1} \left\{ (1 - \varepsilon) \frac{e^{-\gamma}}{M} < n \prod_{i=1}^n |x_i|^{\alpha_{i,n}} < (1 + \varepsilon) e^{-\gamma} \right\} \geq 1 - \frac{1}{n^k}.$$

In the equal weights case we can set $M = 1$, obtaining concentration around $e^{-\gamma} \approx 0.5615$.

Corollary 2.2. *With the same notation as in the preceding theorem, suppose additionally that $\alpha_{i,n} = 1/n$ for $i = 1, \dots, n$. Then there exists an $N = N(k, \varepsilon)$ such that for every $n \geq N$, on \mathbb{S}_1^{n-1} we have*

$$(2.2.1) \quad P_1^{n-1} \left\{ (1 - \varepsilon) e^{-\gamma} < n \prod_{i=1}^n |x_i|^{1/n} < (1 + \varepsilon) e^{-\gamma} \right\} \geq 1 - \frac{1}{n^k}.$$

Remark 2.3. The bounds given in Theorem 2.1 are optimal in the sense that the upper bound for concentration is achieved in the equal weights case, and it is also possible to obtain concentration around values arbitrarily close to $e^{-\gamma}/M$ for every $M \geq 1$, cf. Remark 2.6 below.

Remark 2.4. When studying the concentration of the GM-AM ratios, in addition to the uniform probabilities on the spheres $\mathbb{S}_{a_n}^{n-1}$, there are other reasonable notions of random choice of a vector. Suppose, for instance, that we are in the equal weights case $a_{1,n} = 1$, with $\mathbb{S}_{a_n}^{n-1} = \mathbb{S}_1^{n-1}$. Denote this particular GM-AM ratio by $r_{1,n}$. We might, for example, select points from the whole space \mathbb{R}^n , according the exponential density $2^{-n} e^{-\|x\|_1}$ on \mathbb{R}^n , or the uniform density over $\mathbb{B}_1^n(r)$. Of course, these probabilities give more weight to small vectors than to large ones. But since the GM-AM ratio is homogeneous of degree zero, this makes no difference: The averages of the GM-AM ratio are equal over all spheres $\mathbb{S}_1^{n-1}(r)$ centered at zero with radius $r > 0$, so by the coarea formula, the same answer is obtained when using the exponential probability over \mathbb{R}^n or the uniform probability over $\mathbb{B}_1^n(r)$.

After having proved Corollary 2.2, I came across a related, large sample statistics result in the literature, cf. [Gl, Theorem 5.1], which in the special case $k = 1$ states that $r_{1,n} \rightarrow e^{-\gamma}$ with probability 1 as $n \rightarrow \infty$, where the independent, identically distributed random variables X_i take values in $[0, \infty)$ according to an exponential distribution. Thus, probability refers to the corresponding product probability in the countably infinite product of positive semiaxes. Informally, this result has the same content as Corollary 2.2: If one chooses a large quantity of non-negative numbers at random, then with very high probability the GM-AM ratio will be very close to $e^{-\gamma}$. Formally however, Corollary 2.2 seems to be stronger, since by rewriting it in terms of the exponential distribution, it yields the above result, while the other direction does not follow from the statement of [Gl, Theorem 5.1] (a problem is that when defining a probability on an infinite product many new measure zero sets may appear, which might have large outer measure from the n dimensional viewpoint). The proof of [Gl, Theorem 5.1]

is not provided there, so I do not know whether it might yield something similar to Corollary 2.2.

Concentration may occur around levels different from $e^{-\gamma}$. In fact, given any $t \in (0, e^{-\gamma}]$ it is possible to find a uniformly bounded (by a fixed $M = M(t)$) sequence of weights such that the ratio of the geometric mean over the arithmetic mean concentrates around t . Observe in the next result that $M^{-\frac{M-1}{M+1}}$ maps $\{1 \leq M\}$ onto $(0, 1]$.

Theorem 2.5. *Let $1 \leq M$ and let $L \gg M$. For $n \geq L$ define the sequence $\{a_{i,n} : i = 1, \dots, n\}$ of weights as follows: $a_{j,n} = M$ whenever $j \leq n/(M+1)$, $a_{j,n} = 1/M$ whenever $j \geq 1 + n/(M+1)$, and if j is the least integer strictly larger than $n/(M+1)$, then $a_{j,n} \in [1/M, M]$ is chosen so $\sum_{i=1}^n a_{i,n} = n$. Then there exists an $N = N(k, \varepsilon)$ such that for every $n \geq N$, the GM-AM ratio on $\mathbb{S}_{a_n}^{n-1}$ satisfies*

$$(2.5.1) \quad P_{a_n}^{n-1} \left\{ (1 - \varepsilon) \frac{e^{-\gamma}}{M^{\frac{M-1}{M+1}}} < n \prod_{i=1}^n |x_i|^{\alpha_{i,n}} < (1 + \varepsilon) \frac{e^{-\gamma}}{M^{\frac{M-1}{M+1}}} \right\} \geq 1 - \frac{1}{n^k}.$$

Remark 2.6. Note that $1/M^{\frac{M-1}{M+1}}$ is close to $1/M$ provided M is large. To show optimality of the lower bound in Theorem 2.1 for small values of $M > 1$, we can modify the finite sequences given in the preceding theorem by using $1/M^j$ as the small weights (with $j \gg 1$) instead of $1/M$, and a few more large weights M , in order to add up to n . Then the argument proceeds as in the proof of Theorem 2.5.

Suppose next that the largest weights for the n -th averages are given by a function $f(n) < n$ with $\lim_n f(n) = \infty$. Arguing as in Theorem 2.5 we obtain concentration of the GM-AM ratio around 0.

Theorem 2.7. *Let $f(n) < n$ satisfy $\lim_n f(n) = \infty$, and define $\{a_{i,n} : i = 1, \dots, n\}$ by $a_{j,n} = f(n)$ whenever $j \leq n/(f(n)+1)$, $a_{j,n} = 1/f(n)$ whenever $j \geq 1 + n/(f(n)+1)$, and if j is the least integer strictly larger than $n/(f(n)+1)$, then $a_{j,n} \in [1/f(n), f(n)]$ is chosen so $\sum_{i=1}^n a_{i,n} = n$. Then there exists an $N = N(k, \varepsilon)$ such that for every $n \geq N$, the GM-AM ratio on $\mathbb{S}_{a_n}^{n-1}$ satisfies*

$$(2.7.1) \quad P_{a_n}^{n-1} \left\{ n \prod_{i=1}^n |x_i|^{\alpha_{i,n}} < \varepsilon \right\} \geq 1 - \frac{1}{n^k}.$$

For completeness, we consider the equal weights variant of the AM-GM inequality

$$(2.7.2) \quad \prod_{i=1}^n |y_i|^{1/n} \leq \sqrt{\frac{1}{n} \sum_{i=1}^n y_i^2}$$

studied in [GluMi]. The arguments are essentially the same as in the previous cases, save that now the appropriate notion of random selection is given by the uniform probability P_2^{n-1} on the euclidean unit sphere $\mathbb{S}_2^{n-1} = \{\|y\|_2 = 1\}$ (alternatively one might use, for instance,

the standard gaussian measure on \mathbb{R}^n to obtain the same conclusion, by the reasons given in remark 2.4). On \mathbb{S}_2^{n-1} , $\prod_{i=1}^n |y_i|^{1/n} / \sqrt{\frac{1}{n} \sum_{i=1}^n y_i^2} = \sqrt{n} \prod_{i=1}^n |y_i|^{1/n}$.

Theorem 2.8. *Let $k, \varepsilon > 0$ and let P_2^{n-1} be the uniform probability on \mathbb{S}_2^{n-1} . Then there exists an $N = N(k, \varepsilon)$ such that for every $n \geq N$,*

$$(2.8.1) \quad P_2^{n-1} \left\{ (1 - \varepsilon) \sqrt{2} \exp \left(\frac{\Gamma'(\frac{1}{2})}{2\Gamma(\frac{1}{2})} \right) < \sqrt{n} \prod_{i=1}^n |y_i|^{1/n} < (1 + \varepsilon) \sqrt{2} \exp \left(\frac{\Gamma'(\frac{1}{2})}{2\Gamma(\frac{1}{2})} \right) \right\} \geq 1 - \frac{1}{n^k}.$$

Remark 2.9. $\sqrt{2} \exp \left(\frac{\Gamma'(\frac{1}{2})}{2\Gamma(\frac{1}{2})} \right) \approx 0.5298$.

Remark 2.10. Since $\sqrt{\frac{1}{n} \sum_{i=1}^n y_i^2} \geq \frac{1}{n} \sum_{i=1}^n y_i$ by Jensen's inequality, Theorem 2.8 shows that with arbitrarily high probability

$$(2.10.1) \quad r_{1,n}(x) := \frac{\prod_{i=1}^n |x_i|^{1/n}}{\frac{1}{n} \sum_{i=1}^n |x_i|} \geq (1 - \varepsilon) \sqrt{2} \exp \left(\frac{\Gamma'(\frac{1}{2})}{2\Gamma(\frac{1}{2})} \right)$$

on the *euclidean* unit sphere \mathbb{S}_2^{n-1} , provided n is sufficiently large. Analogous remarks can be made about the behavior of the GM-AM ratio obtained from (2.7.2) on \mathbb{S}_1^{n-1} , by using Corollary 2.2.

Remark 2.11. Often concentration is a consequence of some type of uniform Lipschitz behavior (cf., for instance, Levy concentration theorem in 3 $\frac{1}{2}$.19 p. 142 of [Gro]). This is not the case here. Consider the equal weights ratio $r_{1,n}$, for instance. If we take as our sample space either \mathbb{R}^n or $\mathbb{B}_1^n(r)$, then $r_{1,n}$ is not even continuous, regardless of how $r_{1,n}(0)$ is defined. On \mathbb{S}_1^{n-1} the Lipschitz constant of $r_{1,n}$ depends on n : Set $x = (1/n, 1/n, 1/n, \dots, 1/n)$ and $y = (0, 2/n, 1/n, \dots, 1/n)$. Then $r_{1,n}(x) - r_{1,n}(y) = 1 - 0 = 1$, while $\|x - y\|_1 = 2/n$.

3. LEMMAS AND PROOFS.

Let us recall the coarea formula (for additional information we refer the reader to [Fe], pp. 248-250, or [EG], pp. 117-119):

$$(3.0.1) \quad \int_{\mathbb{R}^n} g(x) |Jf(x)| dx = \int_{\mathbb{R}} \int_{\{f^{-1}(t)\}} g(x) d\mathcal{H}^{n-1}(x) dt.$$

Here f is assumed to be Lipschitz, and $|Jf(x)| := \sqrt{\det df(x) df(x)^t}$ denotes the modulus of the Jacobian. We also remind the reader about some well known facts regarding the Γ function (cf. [Lu], for instance): i) It admits the asymptotic expansion

$$(3.0.2) \quad \Gamma(z) = e^{-z} z^{z-1/2} \sqrt{2\pi} \left(1 + \frac{1}{12z} + O(z^{-2}) \right).$$

ii) $\Gamma(1/2) = \pi^{1/2}$. iii) $\Gamma'(1) = -\gamma$ (Euler's constant). iv) The logarithmic derivative of the Γ function is an analytic function on $\mathbb{C} \setminus \{0, -1, -2, \dots\}$; in particular, it is continuous there.

The following notation, introduced in the preceding section, is recalled here: $\alpha_{i,n} > 0$, $\sum_{i=1}^n \alpha_{i,n} = 1$, and $a_{i,n} = n\alpha_{i,n}$. Without loss of generality we assume that the weights are arranged in decreasing order: $a_{1,n} \geq a_{2,n} \geq \dots \geq a_{n,n}$. The associated unit sphere and the uniform probability on it are denoted by $\mathbb{S}_{a_n}^{n-1}$ and $P_{a_n}^{n-1}$ respectively, and the norm, by $\|\cdot\|_{a_n}$.

Lemma 3.1. *Let $s \in \mathbb{R}$ satisfy $1 + sa_{1,n} > 0$. Then the expectation of $\prod_{i=1}^n |x_i|^{a_{i,n}s}$ over $\mathbb{S}_{a_n}^{n-1}$ is given by*

$$(3.1.1) \quad E \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right) := \int_{\mathbb{S}_{a_n}^{n-1}} \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right) dP_{a_n}^{n-1}(x) = \frac{\Gamma(n)}{\Gamma((1+s)n)} \left(\prod_{i=1}^n \frac{\Gamma(1+a_{i,n}s)}{a_{i,n}^{a_{i,n}s}} \right).$$

By renormalization of the weights, $a_{1,n} \geq 1$. The restriction $s > -1/a_{1,n}$ is of no consequence to us since we will be using this result for $s < 0$ (to obtain lower bounds) when $\alpha_{1,n} \leq M$ and $s \approx 0$ (but $s \gg 1/n$).

Proof. It is well known and easy to check that the volume of the $(\mathbb{R}^n, \|\cdot\|_1)$ -unit ball is $|\mathbb{B}_1^n| = 2^n/n!$. Let T_n be the linear transformation satisfying $T_n(\mathbb{S}_1^{n-1}) = \mathbb{S}_{a_n}^{n-1} = \{\sum_{i=1}^n a_{i,n}|x_i| = 1\}$. Then $\det T_n = \prod_{i=1}^n a_{i,n}^{-1}$. Now for every $x \in \mathbb{R}^n \setminus \cup_{i=1}^n \{x_i \neq 0\}$, the function $f(x) = \|x\|_{a_n}$ is differentiable, and $df(x)df(x)^t$ is a 1×1 matrix, so

$$|Jf(x)| = \sqrt{\det(df(x)df(x)^t)} = \sqrt{\sum_{i=1}^n a_{i,n}^2} = \|a_n\|_2$$

a.e. on \mathbb{R}^n . Set $g(x) = 1/|Jf(x)|$ in (3.0.1), and denote by $\mathbb{S}_{a_n}^{n-1}(\rho)$ the sphere centered at 0 of radius ρ (when $\rho = 1$ we usually omit it). Then

$$\begin{aligned} \frac{2^n}{n! \prod_{i=1}^n a_{i,n}} &= |T_n(\mathbb{B}_1^n)| = |\mathbb{B}_{a_n}^n| = \int_{\mathbb{B}_{a_n}^n} dx = \int_0^1 \int_{\mathbb{S}_{a_n}^{n-1}(\rho)} \frac{1}{\|a_n\|_2} d\mathcal{H}^{n-1}(x) d\rho \\ &= \frac{|\mathbb{S}_{a_n}^{n-1}|}{\|a_n\|_2} \int_0^1 \rho^{n-1} d\rho = \frac{|\mathbb{S}_{a_n}^{n-1}|}{n\|a_n\|_2}, \end{aligned}$$

so $|\mathbb{S}_{a_n}^{n-1}| = 2^n \|a_n\|_2 / (\Gamma(n) \prod_{i=1}^n a_{i,n})$.

Next, set $g(x) = \prod_{i=1}^n |x_i|^{a_{i,n}s} \exp(-\sum_{i=1}^n a_{i,n}|x_i|) / |Jf(x)|$ in (3.0.1). Then

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right) \exp \left(-\sum_{i=1}^n a_{i,n}|x_i| \right) dx \\ &= \frac{1}{\|a_n\|_2} \int_0^\infty e^{-t} \int_{\{\|x\|_{a_n}=t\}} \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right) d\mathcal{H}^{n-1}(x) dt \\ &= \frac{1}{\|a_n\|_2} \int_0^\infty t^{ns} e^{-t} \int_{\{\|x\|_{a_n}=t\}} \left(\prod_{i=1}^n \left| \frac{x_i}{t} \right|^{a_{i,n}s} \right) d\mathcal{H}^{n-1}(x) dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\|a_n\|_2} \int_0^\infty t^{ns} e^{-t} \int_{\{\|x\|_{a_n}=1\}} \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right) t^{n-1} d\mathcal{H}^{n-1}(x) dt \\
&= \frac{1}{\|a_n\|_2} \int_{\mathbb{S}_{a_n}^{n-1}} \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right) dP_1^{n-1}(x) \frac{2^n \|a_n\|_2}{\Gamma(n) \prod_{i=1}^n a_{i,n}} \int_0^\infty t^{(1+s)n-1} e^{-t} dt.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right) \exp \left(- \sum_{i=1}^n a_{i,n} |x_i| \right) dx \\
&= 2^n \prod_{i=1}^n \left(\int_0^\infty t^{a_{i,n}s} e^{-a_{i,n}t} dt \right) = 2^n \prod_{i=1}^n \frac{\Gamma(1 + a_{i,n}s)}{a_{i,n}^{1+s}}
\end{aligned}$$

and

$$\int_0^\infty t^{(1+s)n-1} e^{-t} dt = \Gamma((1+s)n),$$

(3.1.1) follows. \square

Lemma 3.2. Fix $M \geq 1$ and $\varepsilon \in (0, 1)$. Then there exists a $\delta > 0$ such that for all $s \in (-\delta, \delta) \setminus \{0\}$ and every $t \in (0, M]$, we have

$$(3.2.1) \quad (1 - \varepsilon) e^{-t\gamma} < \Gamma(1 + st)^{1/s} < (1 + \varepsilon) e^{-t\gamma}.$$

Proof. For $s \neq 0$ sufficiently close to 0, the term $\Gamma(1 + s)^{1/s}$ can be estimated using L'Hôpital's rule and the continuity of $\Gamma'(x)/\Gamma(x)$ at 1:

$$(3.2.2) \quad \lim_{s \rightarrow 0} \Gamma(1 + s)^{1/s} = \exp \left(\frac{\Gamma'(1)}{\Gamma(1)} \right) = e^{-\gamma}.$$

Thus, we can choose $\delta > 0$ such that for all $s \in (-\delta M, \delta M) \setminus \{0\}$,

$$(3.2.3) \quad (1 - \varepsilon)^{1/M} e^{-\gamma} < \Gamma(1 + s)^{1/s} < (1 + \varepsilon)^{1/M} e^{-\gamma}.$$

Next, let $s \in (-\delta, \delta) \setminus \{0\}$ and pick $t \in (0, M]$. Using the change of variables $s \mapsto st$ in (3.2.3), we get (3.2.1). \square

Lemma 3.3. Suppose there exists an $M \geq 1$ such that for all $n \in \mathbb{N}$, $n \geq 2$, and all $i = 1, \dots, n$, $a_{i,n} \leq M$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $s \in (-\delta, \delta) \setminus \{0\}$ and every $n \in \mathbb{N}$ with $n \geq 2$ we have

$$(3.3.1) \quad (1 - \varepsilon) e^{-\gamma} < \left(\prod_{i=1}^n \Gamma(1 + a_{i,n}s) \right)^{1/sn} < (1 + \varepsilon) e^{-\gamma}.$$

Proof. Fix n . Set $t = a_{i,n}$ in (3.2.1), to get, for each $i = 1, \dots, n$,

$$(1 - \varepsilon) e^{-a_{i,n}\gamma} < \Gamma(1 + sa_{i,n})^{1/s} < (1 + \varepsilon) e^{-a_{i,n}\gamma}.$$

Taking the product over i and using $\sum_{i=1}^n a_{i,n} = n$ we obtain (3.3.1). \square

Lemma 3.4. *For $i = 1, \dots, n$ let $t_i > 0$. Subject to the restriction $\sum_{i=1}^n t_i \geq n$, the function $f(t_1, \dots, t_n) := \prod_{i=1}^n t_i^{t_i}$ achieves its global minimum when $t_1 = t_2 = \dots = t_n = 1$, where it takes the value 1.*

Proof. Setting $t_i^{t_i} = 1$ (by continuity) when $t_i = 0$, the function $f(t_1, \dots, t_n) := \prod_{i=1}^n t_i^{t_i}$ extends to the set where $t_i \geq 0$ for $i = 1, \dots, n$. Using Lagrange multipliers when all $t_i > 0$ and induction when one or several of the t_i equal 0, the result follows. \square

Proof of Theorem 2.1. Fix $k > 0$ and $\varepsilon \in (0, 1)$ (k denotes a large parameter and ε a small one). To obtain an upper bound for $r_{1,n}(x) = n \prod_{i=1}^n |x_i|^{\alpha_{i,n}}$ on \mathbb{S}_{an}^{n-1} , we use Chebyshev's inequality

$$(3.4.1) \quad P_{an}^{n-1} \left\{ \prod_{i=1}^n |x_i|^{a_{i,n}s} \geq t \right\} \leq \frac{1}{t} E \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right)$$

with $s = s(k, \varepsilon) > 0$ selected very close to 0, but fixed (so eventually $s \gg n^{-1}$). Set

$$(3.4.2) \quad t = 2n^k E \left(\prod_{i=1}^n |x_i|^{a_{i,n}s} \right).$$

It follows from (3.4.1) that

$$(3.4.3) \quad P_{an}^{n-1} \left\{ \prod_{i=1}^n |x_i|^{\alpha_{i,n}} < t^{1/sn} \right\} \geq 1 - \frac{1}{2n^k},$$

so it suffices to prove

$$(3.4.4) \quad nt^{1/s_0n} < (1 + \varepsilon)e^{-\gamma}$$

for some small s_0 (to be chosen below) and all n large enough; in particular, we always assume that $s \ll 1$, and that $n \gg s^{-1}$ whenever both n and s appear in the same expression. Pick $\delta = \delta(\varepsilon) > 0$ such that $(1 + \delta)^3 < 1 + \varepsilon$. By the choice of t (in (3.4.2)) and Lemma 3.1, we have

$$(3.4.5) \quad t^{1/sn} = (2n^k)^{1/sn} \left(\frac{\Gamma(n)}{\Gamma((1+s)n)} \right)^{1/sn} \left(\prod_{i=1}^n \frac{\Gamma(1 + a_{i,n}s)}{a_{i,n}^{a_{i,n}s}} \right)^{1/sn}.$$

Next we bound each of the three factors in the right hand side of (3). From Lemmas 3.3 and 3.4 we get, for all $s > 0$ sufficiently small,

$$(3.4.6) \quad \left(\prod_{i=1}^n \frac{\Gamma(1 + a_{i,n}s)}{a_{i,n}^{a_{i,n}s}} \right)^{1/sn} \leq (1 + \delta)e^{-\gamma}.$$

From the asymptotic expansion (3.0.2) of Γ and the assumption $n \gg s^{-1}$ we obtain

$$(3.4.7) \quad \left(\frac{\Gamma(n)}{\Gamma((1+s)n)} \right)^{1/sn} \leq \left(\frac{e}{n} \right) \left(\frac{(1+s)^{1/2sn}}{(1+s)^{1+1/s}} \right) \left(1 + O\left(\frac{1}{n}\right) \right)^{1/2sn} \leq \left(\frac{1}{n} \right) \left(\frac{e}{(1+s)^{1/s}} \right).$$

Finally, by L'Hôpital's rule, given any $s > 0$ we have

$$(3.4.8) \quad \lim_{n \rightarrow \infty} (2n^k)^{1/sn} = 1.$$

Thus, we can select $s_0 = s_0(\varepsilon) > 0$ so small in (3.4.7) that $e/(1+s_0)^{1/s_0} < 1 + \delta$. Choosing $N = N(k, s_0(\varepsilon))$ such that for all $n \geq N$ we have $(2n^k)^{1/s_0 n} < 1 + \delta$, inequality (3.4.4) follows.

Observe that the hypothesis $a_{i,n} \leq M$ on the renormalized weights entails that their geometric mean $\prod_{i=1}^n a_{i,n}^{a_{i,n}}$ is also bounded above by M . To obtain

$$P_1^{n-1} \left\{ (1 - \varepsilon) \frac{e^{-\gamma}}{M} < n \prod_{i=1}^n |x_i|^{\alpha_{i,n}} \right\} \geq 1 - \frac{1}{2n^k},$$

basically all we need to do is to follow the same steps as before, but using $s < 0$ instead of $s > 0$, and the preceding observation instead of Lemma 3.4. So we avoid the repetition. \square

The preceding proof works by respectively giving upper and lower bounds for $nt^{1/sn}$ when $s > 0$ and $s < 0$ are close enough to zero. Since

$$nt^{1/sn} = (2n^k)^{1/sn} \left[n \left(\frac{\Gamma(n)}{\Gamma((1+s)n)} \right)^{1/sn} \right] \left(\prod_{i=1}^n \frac{\Gamma(1+a_{i,n}s)}{a_{i,n}^{a_{i,n}s}} \right)^{1/sn}$$

and the first two factors on the right hand side approach 1 as $n \rightarrow \infty$, concentration is controlled by the third factor. Furthermore, by Lemma 3.3, for all s sufficiently close to zero we have

$$(3.4.9) \quad (1 - \varepsilon)e^{-\gamma} < \left(\prod_{i=1}^n \Gamma(1 + a_{i,n}s) \right)^{1/sn} < (1 + \varepsilon)e^{-\gamma},$$

so in order to determine how

$$\left(\prod_{i=1}^n \frac{\Gamma(1 + a_{i,n}s)}{a_{i,n}^{a_{i,n}s}} \right)^{1/sn}$$

behaves it is enough to estimate its denominator. The concentration results in Theorems 2.5 and 2.7 are obtained by giving sequences of weights for which the behavior of $\prod_{i=1}^n a_{i,n}^{a_{i,n}/n}$ is easily understood.

Proof of Theorem 2.5. Denote by j the integer part of $n/(M+1)$. Suppose first that $n/(M+1)$ is an integer. Then the sequence of weights $a_{i,n} = M$ for $i \leq j$ and $a_{i,n} = 1/M$ for $i > j$ satisfies $\sum_{i=0}^n a_{i,n} = n$. If $n/(M+1)$ is not an integer, then $jM + (n-j)/M < n$ while $(j+1)M + (n-j-1)/M > n$. Thus, there exists a $t \in [1/M, M]$ such that redefining $a_{j+1,n} = t$ (instead of $1/M$) we have $\sum_{i=0}^n a_{i,n} = n$. Since for this family of weights $\lim_{n \rightarrow \infty} \prod_{i=1}^n a_{i,n}^{a_{i,n}/n} = M^{\frac{M-1}{M+1}}$, the result follows by using the same argument as in the proof of Theorem 2.1. \square

Theorem 2.7 is proven in the same way as the previous one, and Theorem 2.8, as Theorem 2.1, save for the fact that we use the uniform probability P_2^{n-1} on the euclidean unit sphere,

and the corresponding expectation of $\prod_{i=1}^n |y_i|^s$ over \mathbb{S}_2^{n-1} . This expectation is computed within the proof of Proposition 1 in [GluMi]:

$$(3.4.10) \quad E \left(\prod_{i=1}^n |y_i|^s \right) := \int_{\mathbb{S}_2^{n-1}} \left(\prod_{i=1}^n |y_i|^s \right) dP_2^{n-1} = \left(\frac{\Gamma \left(\frac{1+s}{2} \right)}{\Gamma \left(\frac{1}{2} \right)} \right)^n \frac{\Gamma \left(\frac{n}{2} \right)}{\Gamma \left(\frac{1+s}{2} n \right)}.$$

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